§1 Cartier duality S scheme,

$G = \underline{\mathrm{Spec}}_{\mathcal{O}_S} A$ finite loc free commutative S-group scheme

A comes with

$$\left.\begin{array}{ll} 1 : \mathcal{O}_S \to A & e^* : A \to \mathcal{O}_S \\ m : A \otimes_{\mathcal{O}_S} A \to A & \mu^* : A \to A \otimes_{\mathcal{O}_S} A \end{array}\right\} \circledast$$

& both m, μ^* are associative, commutative.

<u>Def</u> <u>Cartier dual</u> of G $\overline{\overline{\text{def}}}$ the finite loc free commutative group scheme $\hat{G}$ obtained by applying

$\mathrm{Hom}_{\mathcal{O}_S}(-, \mathcal{O}_S)$ to $\circledast$

<u>Example</u>

1) $\widehat{\underline{\mathbb{Z}/n\mathbb{Z}}_S} = \mu_{n,S}$, $\hat{\mu}_{n,S} = \underline{\mathbb{Z}/n\mathbb{Z}}_S$

This <u>always</u> holds, also if $n \notin \mathcal{O}_S^\times$.

2) $p \cdot \mathcal{O}_S = 0$. $\alpha_p := \ker \left(F_{\mathbb{G}_a/S} : \mathbb{G}_a \to \mathbb{G}_a \atop f \mapsto f^p \right) = \mathrm{Spec}_{\mathcal{O}_S} \mathcal{O}_S[t]/t^p$

Then $\hat{\alpha}_p = \alpha_p$.

Prop $\hat{G}$ represents the functor

$$T/S \longmapsto \mathrm{Hom}_{T\text{-Grp Sch}}(T \times_S G, \mathbb{G}_{m,T})$$

Proof Say $T = \mathrm{Spec}\, R' \longrightarrow S = \mathrm{Spec}\, R$

$$\begin{aligned} \hat{G}(T) &= \mathrm{Hom}_{R\text{-alg}}(A^\vee, R') \\ &= \mathrm{Hom}_{R'\text{-alg}}(R' \otimes_R A^\vee, R') \\ &\subseteq \mathrm{Hom}_{R'\text{-mod}}(R' \otimes_R A^\vee, R') \\ &= R' \otimes_R (A^\vee)^\vee = R' \otimes_R A. \end{aligned}$$

Also $\mathrm{Hom}(G_{R'}, \mathbb{G}_{m,R'}) = \mathrm{Hom}_{R'\text{-bi-algebra}}(R'[t^{\pm 1}], R' \otimes_R A)$

$$\subseteq R' \otimes_R A.$$

$$\varphi \longmapsto \varphi(t)$$

Image of last inclusion:

$$\{\alpha \in (R' \otimes_R A)^\times \mid \mu^*(\alpha) = \alpha \otimes \alpha\}$$

$$= \{\alpha \in R' \otimes_R A \mid \mu^*(\alpha) = \alpha \otimes \alpha,\ e^*(\alpha) = 1\}$$

<u>Claim</u>

Proof If $\mu^*(\alpha) = \alpha \otimes \alpha$, then $\underset{\underset{e^*(\alpha)}{\parallel}}{\mu^*(e^*(\alpha))} = e^*(\alpha) \otimes e^*(\alpha)$
$= e^*(\alpha)^2$

So $e^*(\alpha)$ is idempotent.

If α unit, $e^*(\alpha)$ unit, hence $= 1$.

For converse direction, one does a fiber wise argument

i.e. assumes $R = k$ alg closed field.

Then $\pi_0(G) = G/G^\circ = \underline{\Gamma}_k$ constant group scheme.

$e^*(\alpha)$ unit $\Rightarrow$ $\mu^*(\alpha)(\gamma, \gamma^{-1})$ unit $\forall \gamma \in \Gamma$.

But also $\mu^*(\alpha)(\gamma, \gamma^{-1}) = \alpha(\gamma) \cdot \alpha(\gamma^{-1})$, so

$\alpha(\gamma)$ a unit $\forall \gamma$.

Since also $\pi_0(G) \hookrightarrow G$ (k alg closed)

is closed subgroup scheme, equal to G_{red},

α is unit. $\square$

Now $\mu^*(\alpha) = \alpha \otimes \alpha$, $e^*(\alpha) = 1$ are precisely the

conditions on $\alpha : R' \otimes_R A^\vee \to R'$ to be a

alg map. $\square$

§2 Application to abelian varieties

<u>Cor</u> $n \in \mathcal{O}_S^\times$, E/S EC. Then

$$\hat{E}[n] \xrightarrow[e_n]{\cong} \widehat{E[n]}.$$

<u>Proof</u> Both sides finite étale grp sch and e_n fiberwise non-degenerate. $\square$

<u>Recall</u> Abelian var over S $\underset{\text{def}}{=}$ smooth, proper group schem $X \to S$ w/ conn fibers.

Isogeny $X \to Y$ of ab. vars. $\underset{\text{def}}{=}$ group scheme map that is finite + loc. free

Dual AV $\hat{X}$ of X $\underset{\text{def}}{=}$ represents Pic°_X = translation-invariant line bundles

<u>Thm</u> X/S ab var, $\lambda: X \to Y$ isogeny. $\hat{\lambda}: \hat{Y} \to \hat{X}$ dual isogeny. Then

$$\ker \hat{\lambda} \xrightarrow{\cong} \widehat{\ker \lambda} \quad \text{canonically.}$$

<u>Proof</u> See Mumford §15. $\square$

Definition of $\ker(\lambda) \times \ker(\hat{\lambda}) \xrightarrow{e^\lambda} \mathbb{G}_m$

$x \in \ker(\lambda)(T) \quad L \in \ker(\hat{\lambda})(T) \subseteq \mathrm{Pic}(T \times_S Y) / p_T^* \mathrm{Pic}(T)$.

Write $S = T$ in following.

Locally on S, pick $\psi : \mathcal{O}_X \xrightarrow{\cong} \lambda^*(L)$

Consider

$$\psi_x : \mathcal{O}_X \underset{\cong}{\xrightarrow{t_x^*}} t_x^* \mathcal{O}_X \underset{\cong}{\xrightarrow{t_x^* \lambda^*(L)}} t_x^* \lambda^*(L) = \lambda^*(L)$$

put $\quad e^\lambda(x, L) := \psi_x / \psi$.

Example $\operatorname{char} k = p$, $k = \bar{k}$, E/k EC.

Consider $F_{E/k} : \operatorname{Spec} k \times_{F, \operatorname{Spec} k} E =: E^{(p)} \longrightarrow E$.

As scheme $\quad K = \ker(F_{E/k}) \cong \operatorname{Spec} \mathcal{O}_{E^{(p)}} / \mathfrak{m}_e^p$
$$\cong \operatorname{Spec} k[\varepsilon]/\varepsilon^p.$$

Classification of order-p commutat. grp sch / alg closed fields

$\implies K \cong \mu_p$ or α_p.

The dual isogeny $\hat{E} \xrightarrow{\hat{F}_{E|k}} \widehat{E^{(p)}}$ fits into a diagram

$$\begin{tikzcd} E \arrow[r, "p \cdot F_{E|k}^{-1}"] \arrow[d, "\cong"'] & E^{(p)} \arrow[d, "\cong"] \\ \hat{E} \arrow[r, "\hat{F}_{E|k}"] & \widehat{E^{(p)}} \end{tikzcd}$$

(cf. AV §12)

So $E \begin{Bmatrix} \text{supersing} \\ \text{ordinary} \end{Bmatrix} \iff p \cdot F_{E|k}^{-1} \begin{Bmatrix} \text{inseparable} \\ \text{étale} \end{Bmatrix}$

$\iff \hat{F}_{E|k} \begin{Bmatrix} \text{insep.} \\ \text{étale} \end{Bmatrix}$

$p \cdot F_{E|k}^{-1} = \hat{F}_{E|k}$

$\iff \hat{K} = \begin{Bmatrix} \text{infinitesimal} \\ \text{étale} \end{Bmatrix}$

Cartier duality

$\iff K \cong \begin{Bmatrix} \alpha_p \\ \mu_p \end{Bmatrix}$

Clarification

This is a new piece of information!

Before, we were able to distinguish only by looking at all of $E[p]$.